Modeling of Localized Damage and Fracture in Quasibrittle Materials

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Abstract. This paper focuses on strain and damage localization due to propagation and coalescence of microcracks in quasibrittle materials such as concrete or rock. The mathematical description and numerical simulation of such phenomena can be based on an indirect representation of the cracking-induced deformation by inelastic strain, or on a direct representation by discontinuities in the displacement field. The first part of the paper gives a general overview and classification of modeling approaches and numerical techniques related to both types of representations and discusses their possible combination. The second part proposes a new technique for computational resolution of localization zones in regularized softening continua, based on special enrichments of the standard finite element approximations.

1 Representation of Localized Deformation

The behavior of quasibrittle materials (such as concrete, rock, tough ceramics, or ice) subjected to a high level of mechanical solicitations is characterized by the localization of strain and damage in relatively narrow zones and by a gradual development of macroscopic stress-free cracks. Despite a considerable progress in the past two decades, theoretical modeling and computational resolution of the localization process up to structural failure still remains a challenging issue of contemporary solid mechanics.

Narrow zones of highly concentrated evolving microdefects can be represented in many different ways. To establish a systematic classification, we will look at the (i) kinematic description, (ii) material models, and (iii) numerical approximation techniques. From the discussion it will transpire that all these aspects are closely related and a successful modeling approach must combine them in an appropriate manner.

1.1 Kinematic Description

Depending on the regularity of the displacement field, $\boldsymbol{u}(\boldsymbol{x})$, we can distinguish three types of kinematic descriptions. The first one incorporates *strong discontinuities*, i.e., jumps in displacements across a discontinuity curve (in two dimensions) or discontinuity surface (in three dimensions). The strain field, $\boldsymbol{\varepsilon}(\boldsymbol{x})$, then consists of a regular part, obtained by standard differentiation of the displacement field, and a singular part, having the character of a multiple of the Dirac

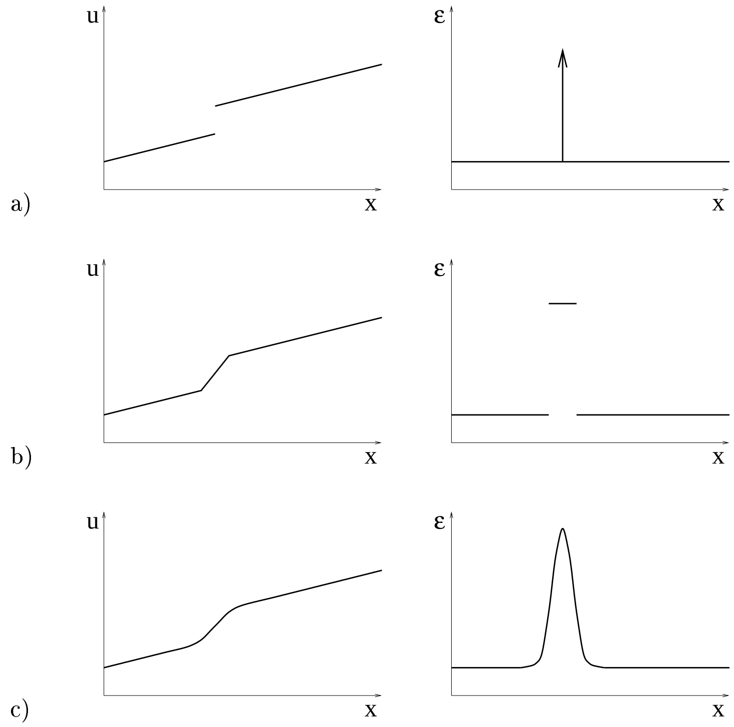

Fig. 1. Kinematic description with a) one strong discontinuity, b) two weak discontinuities, c) no discontinuities

delta distribution. This is schematically shown for the one-dimensional case in Fig. 1a. In physical terms, the strong discontinuity corresponds to a sharp crack.

Another possible kinematic description represents the region of localized deformation by a band of a small but finite thickness, separated from the remaining part of the body by two *weak discontinuities*, i.e., curves or surfaces across which certain strain components have a jump but the displacement field remains continuous. This is illustrated in Fig. 1b. Since the displacement is continuous, the strain components in the plane tangential to the discontinuity surface must remain continuous as well, and only the out-of-plane components can have a jump. In physical terms, the band between the weak discontinuities corresponds to a damage process zone with an almost constant density of microdefects.

Finally, the most regular description uses a continuously differentiable displacement field, and the strain field remains *continuous*. Strain localization is manifested by high strains in a narrow band, with a continuous transition to much lower strains in the surrounding parts of the body. A typical strain profile of this type is shown in Fig. 1c. In physical terms, this corresponds to a damage process zone with a higher concentration of defects around its center.

1.2 Constitutive Models

Each of the kinematic descriptions discussed in the preceding section requires a different type of constitutive law for the cracking material. Of course, one could also consider the material law as the primary component of the model, and say that the regularity of the displacement field obtained as the solution of the corresponding boundary value problem depends on the choice of that law.

In the simplest case, the strong discontinuity can be considered as stress-free, which necessarily leads to a stress singularity at the discontinuity front (crack tip). The criterion for crack propagation is then formulated in terms of the stress intensity factors characterizing the singular part of the stress field. This is the linear elastic fracture mechanics approach, applicable at very large scales, when the process zone is negligible with respect to the characteristic dimensions of the body (such as the crack length and ligament size).

With the exception of huge quasibrittle structures such as concrete dams, it is usually necessary to take into account the finite size of the process zone at least in the direction tangential to the discontinuity surface. In the direction normal to the discontinuity, all inelastic effects are lumped and replaced by the equivalent displacement jump (crack opening), $[[\boldsymbol{u}]]$. Before the growing microdefects coalesce and form a true physical crack, the process zone still carries some stresses, which are represented in the model as cohesive tractions transmitted by the discontinuity. The usual assumption of *cohesive crack models* or *cohesive zone models* is that the cohesive traction depends only on the displacement jump (separation vector), even though the evolution of the process zone is in general affected also by strains (or stresses) in the tangential plane. This means that the strong discontinuity is governed by its own constitutive law, formulated as a traction-separation law, which complements the stress-strain law valid for the continuous part of the body; see Fig. 2a. The area under the traction-separation curve is the energy needed to create a stress-free crack of a unit area, and it is usually called the Mode-I fracture energy, G_F. Even though it is questionable whether this is a true material property, it is certainly a fundamental parameter of every cohesive crack model.

Models with localization bands bounded by weak discontinuities can be considered as simple regularizations of models with strong discontinuities. Instead of lumping all the inelastic effects into a surface, it is possible to distribute them uniformly across the width of a band of a finite thickness h. This naturally leads to the *smeared crack models*, which transform the traction-separation law into a law that links the stress transmitted by the localization band to the average inelastic strain in that band; see Fig. 2b. If $[[u]]$ is the normal component of the displacement jump, the corresponding inelastic normal strain is obviously $\varepsilon_i = [[u]]/h$, and the area under the softening curve, G_F/h, has now the meaning of energy spent per unit volume of the localization band. Instead of splitting the constitutive law into the elastic and inelastic parts (more precisely, into the basic part and the part activated after localization), one could use a law that directly links the stress to the total strain, as is the case, e.g., in continuum dam-

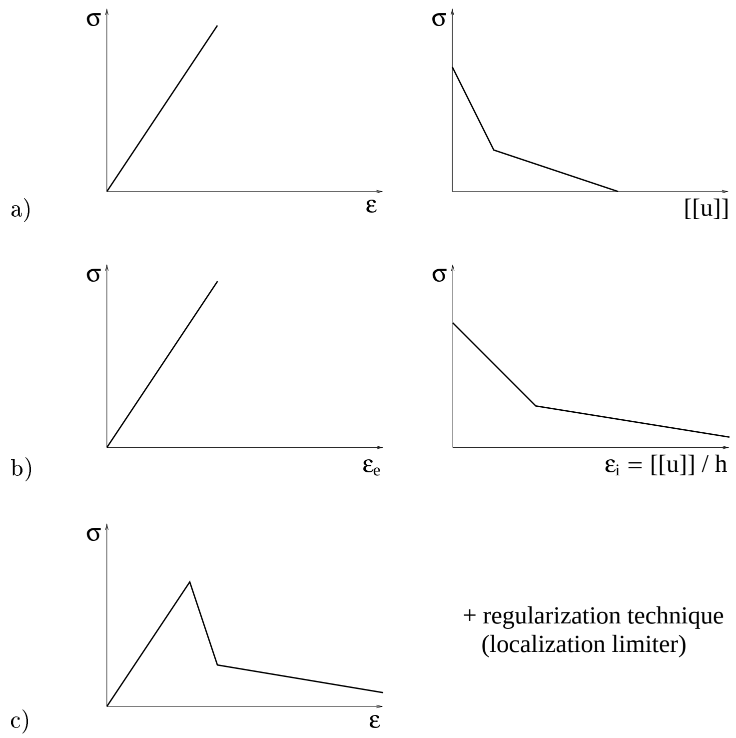

Fig. 2. Schematic representation of the constitutive description: a) cohesive crack model, b) smeared crack model, c) regularized continuum model with softening

age mechanics. The adjustment of the fracture energy is then more complicated but still possible.

Strain fields that remain continuous even after the onset of localization can be obtained with more sophisticated regularization techniques that serve as localization limiters. They are usually based on various forms of *enriched continuum theories*, e.g., on nonlocal or higher-order gradient continua. Such enrichments typically introduce a parameter defining a characteristic length of the material and related to the size and spacing of heterogeneities that control the width of the localized zone. The standard "local" form of the constitutive model corresponds to a deformation process during which the strain field remains uniform. However, the actual solution is nonuniform (either from the very beginning, or at least after bifurcation from a uniform solution), and the relation between the stress and local strain does not follow the same curve at every point. The energy needed to create a completely stress-free process zone depends on the area under the local stress-strain curve and on the characteristic length introduced by the localization limiter. One could define an equivalent dissipation length, l_d, equal to the thickness of a localization band that would give the same energy dissipation when the corresponding local model is used. The dissipation length is roughly proportional to the characteristic length but is also affected by the ductility of the constitutive law and by specific details of the nonlocal formulation; see [1].

1.3 Numerical Approximations

As is clear from the foregoing discussion, each class of constitutive models gives rise to solutions that are characterized by a certain degree of regularity of the kinematic fields (displacements and strains). This must be taken into account by the numerical technique used for an approximate solution of the boundary value problem. We will focus here on the finite element method, with standard as well as enriched finite elements. The idea common to all types of models is that the details of the kinematic representation of the localized process zone can be resolved either by standard finite elements, which often requires fine meshes or frequent remeshing, or by special enrichments used in conjunction with a relatively coarse mesh.

For example, a strong discontinuity can be captured by *interface elements* inserted between two- or three-dimensional elements into which the body is divided; see Fig. 3a left. This is relatively straightforward if the discontinuity curve or surface is known in advance and is taken into account by the mesh generation procedure. In a general case, however, the discontinuity propagates across the body along an unknown path, and a good approximation of that path by interfaces between elements in a fixed mesh is not always possible. One solution is frequent remeshing, at least in the vicinity of the propagating discontinuity tip. As an elegant alternative, the recently emerged *elements with embedded strong discontinuities* [2–4] allow capturing the displacement jump across a segment that has an arbitrary position and orientation with respect to the basic finite element; see Fig. 3a right. Such formulations are based on enrichments of the standard shape functions by special discontinuous functions.

A similar discussion applies to models with localization bands of a finite thickness (Fig. 3b). They can be approximated by *standard finite elements* (this time only by the basic mesh, without the need for introducing interface elements), but if the band is not aligned with a layer of elements, it must be approximated by a zig-zag band. The thickness of the band, h, is dictated by the size of the finite elements, and the softening part of the stress-strain law must be adjusted accordingly, as explained in Section 1.2. The solution is usually biased by the mesh orientation and, for many constitutive models, the misalignment between the band direction and the mesh lines generates spurious stresses (this pathological behavior is sometimes called the stress locking). Again, special *elements with embedded localization bands* [5,6] permit the modeling of bands of an arbitrary direction with respect to the basic mesh, and also of an arbitrary thickness. So, an additional advantage of this approach is that the thickness of the computationally resolved localization band does not depend on the element size any more, and it can be considered as a part of the constitutive model.

Finally, regularized models lead to localized strain profiles that remain continuous. To capture steep strain gradients with standard approximation techniques,

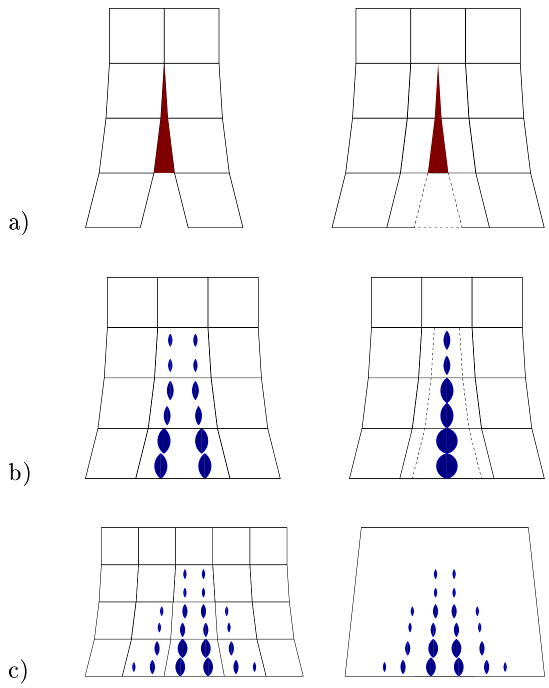

Fig. 3. Finite element models of the a) strong discontinuity, b) localization band, c) localization zone in a regularized softening continuum

it is necessary to use a sufficiently fine discretization (Fig. 3c left). Alternatively, one can develop special enrichments such as the spectral overlays proposed in [7,8] for shear bands modeled by softening viscoplasticity. One of such techniques, specifically designed for the resolution of evolving process zones in quasibrittle materials modeled by nonlocal damage theories, will be examined in Section 2. As shown in the right part of Fig. 3c, the finite elements can then be larger than the thickness of the process zone, which makes the computation much more efficient.

1.4 Combined continuous-discontinuous description

Enhanced formulations dealing with displacement or strain discontinuities embedded in finite elements are potentially very powerful but, in order to avoid pathological locking effects, the kinematic description of such elements and the internal equilibrium conditions must be constructed carefully, and appropriate criteria for the positioning of newly initiated discontinuities must be used. Recently it has been shown [9] that the mesh-induced directional bias can be dramatically reduced by models combining the discontinuous description with a continuous one. The early stage of cracking (at each material point) is represented by a stress-strain law with softening, either in the local form (with adjustment

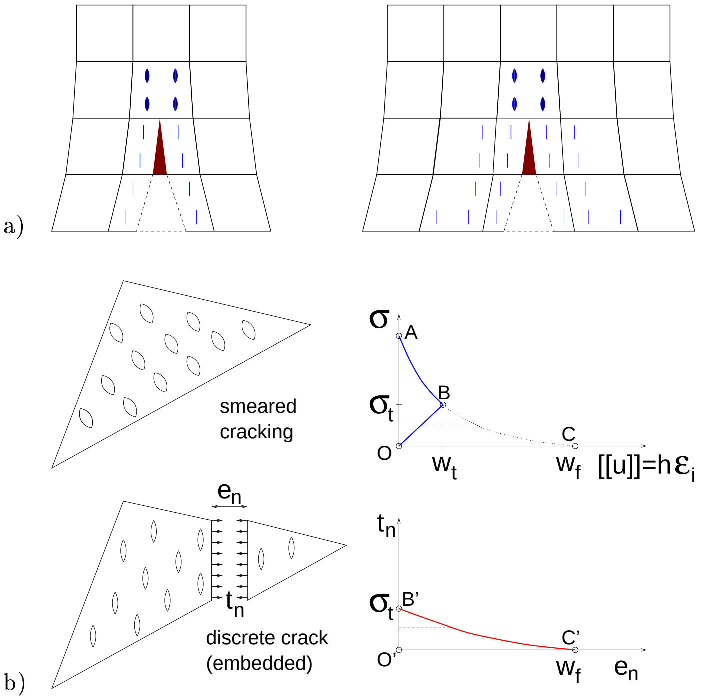

Fig. 4. Combined description: a) kinematic representation, b) constitutive model

of the softening curve according to the element size), or regularized by a localization limiter. When the damage at some point reaches a certain critical level, a discrete crack is introduced and modeled as a strong discontinuity embedded in the corresponding finite element; see Fig. 4a. The material laws describing the continuous and discrete parts of the model must be matched such that the desired global behavior is obtained; see Fig. 4b.

A related idea of a transition from an embedded localization band to an embedded strong discontinuity was explored in [10].

2 Elements with Embedded Localization Zones

2.1 Motivation

The overview of numerical approximation techniques in Section 1.3 has shown that, in analogy to finite elements with embedded discontinuity lines or bands, it may be worthwhile to develop elements with special enrichments that are tailored for regularized continuum models. This idea is illustrated in Fig. 5, which shows from top to bottom three stages of a typical evolution of strain in the direction perpendicular to a propagating process zone. The left column corresponds to the development of the "exact" strain field, from a uniform profile before the onset

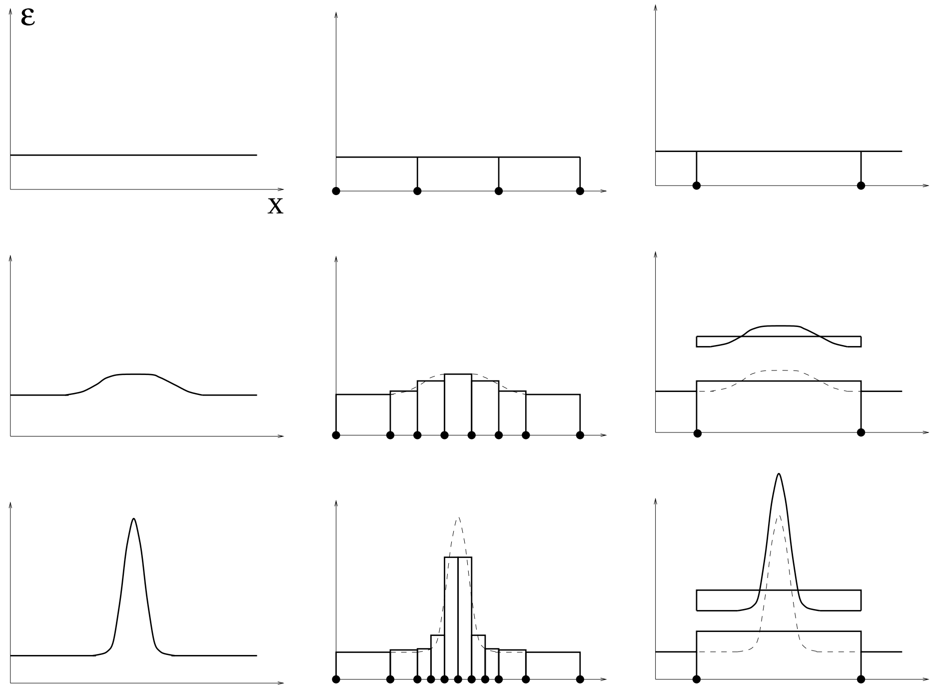

Fig. 5. Evolving strain profile and its approximation using standard low-order finite elements or embedded localization zones

of localization to a highly localized one at later stages of the loading process. The middle column in Fig. 5 illustrates the resolution of the evolving strain profile with low-order (constant-strain) finite elements. If the initial mesh is relatively coarse with respect to the characteristic material length, an adaptive refinement of the mesh is needed as the localization profile evolves. Even then, the approximation is not smooth and can substantially deviate from the actual profile, unless an extremely fine mesh is used. The right column shows that a good resolution can be hoped for if the standard finite element approximation on a coarse mesh is combined with special wave-like enrichments that are adaptively adjusted to the characteristic "wave length" of the evolving localization profile.

An additional argument motivating the development of special elements for regularized continuum models is that, after the onset of localization, standard low-order elements are often incapable of reproducing the state of constant stress. Of course, when linear elements with one integration point are used in a one-dimensional localization problem, the stress profile always remains perfectly uniform. However, this is rather an exception. In multiple dimensions, this would be the case only for underintegrated bilinear elements of a rectangular shape, perfectly aligned with the localization zone. As soon as several integration points per element are used, or the elements have an arbitrary shape and orientation, stress oscillations arise. For nonlocal integral models, the problem is caused by the imbalanced quality of approximations of the local and nonlocal strains. The local strain approximations are discontinuous across element interfaces, while the nonlocal strain approximations are constructed using a discretized integral operator and thus are by definition continuous. A similar problem arises for gradient-type models discretized by mixed finite elements [11], if the approximations of the displacement field and of the additional Lagrange-multiplier field are of the same order.

2.2 Low-Order Elements

To illustrate the stress oscillations, Fig. 6 presents the strain and stress profiles obtained with a one-dimensional version of the nonlocal damage model [12] using linear finite elements with two integration points per element. As already explained, one-dimensional elements with one integration point would give perfectly uniform stress profiles, but in multiple dimensions this is no longer the case, even with constant-strain elements using one-point integration. One-dimensional elements with two-point integration indicate what would happen for bilinear elements with the usual 2×2 integration, so the problem documented here is not artificially caused by an unusual integration scheme. To simplify the plotting procedure, the profiles were constructed by connecting the values at integration points by straight segments, which gives the false impression of continuity. In reality, the strain approximation is piecewise constant, and the stress approximation varies continuously inside each element, with jumps at nodes that connect the elements. The figure presents the results for 4 different meshes and two different states—one just before the onset of localization and the other after softening to about 80% of the peak load. The spatial coordinate x on the horizontal axis is normalized by the characteristic length of the nonlocal continuum,

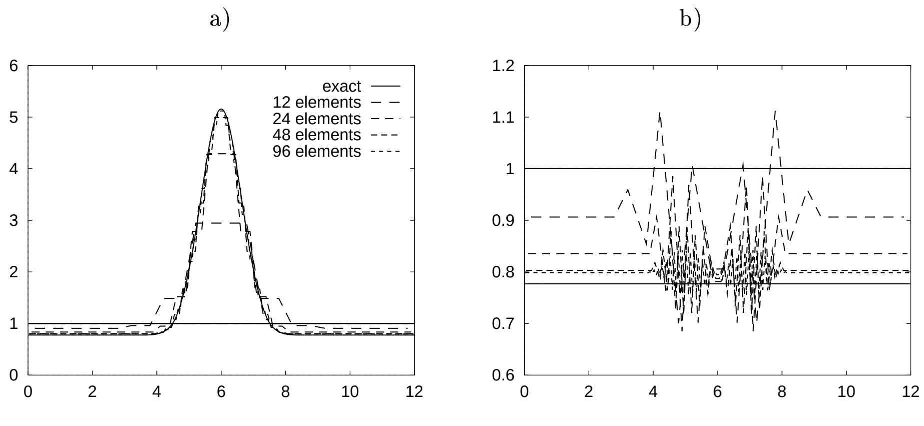

Fig. 6. Localization in a one-dimensional nonlocal continuum approximated by low-order finite elements: a) strain profiles, b) stress profiles

and the stresses and strains on the vertical axes are normalized by their values at the peak of the load-displacement diagram. The profiles before localization are uniform and they are reproduced exactly by each of the meshes. For the localized state we can see that the strain profiles converge to the exact solution as the mesh is refined. The stress profiles also converge, but they exhibit wild oscillations, even on a fine mesh with more than 30 elements across the thickness of the localized zone (the material in the left and right thirds of the bar is unloading elastically). The weak form of the equilibrium conditions implies that the average stress in each element must be the same, but even this average stress level is still quite far from the exact solution. Note that on the coarser meshes the stresses at some integration points are above the peak stress.

2.3 Higher-Order Elements

At first one may think that the simplest remedy is to increase the polynomial order of the finite element approximation. Higher-order polynomials should better correspond to the smooth character of the exact solution and reduce the imbalance between the quality of the local and nonlocal strain approximations. This is only partially true. If the mesh is sufficiently fine, higher-order elements give indeed better results than linear ones (of course, a fair comparison must be done for the same number of nodes). However, if the basic mesh is relatively coarse compared to the characteristic material length, it is not sufficient to increase the order of polynomial approximation, keeping the element size constant. Fig. 7 shows that the stress oscillations remain large even for a quartic displacement interpolation. Only three finite elements across the entire bar length were used, with a large number of integration points (16 in the middle element and 6 in each of the other elements). The results are somewhat disappointing, because the size and position of the middle element matches very well with the size and

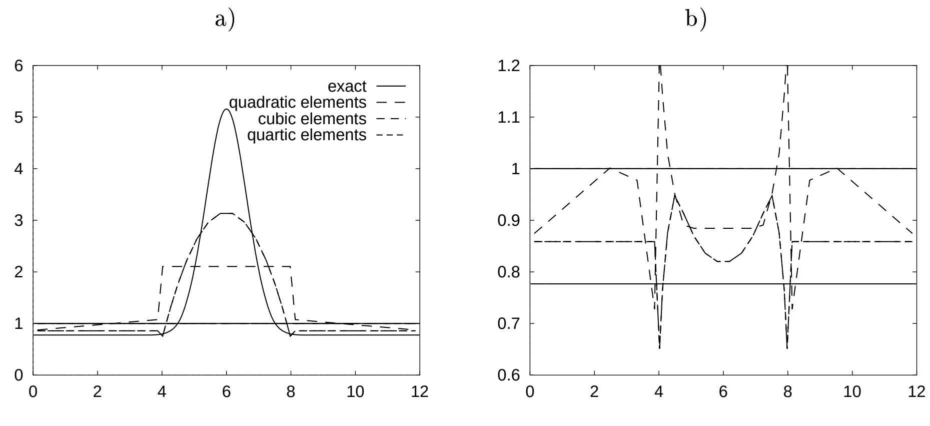

Fig. 7. Localization in a one-dimensional nonlocal continuum approximated by higher-order finite elements: a) strain profiles, b) stress profiles

position of the actual localization zone, so the element has in principle a good chance to closely approximate the exact solution. Note that, due to symmetry of the present problem, there is no difference between the quartic and the cubic interpolation.

2.4 Enriched Elements

Finally, Fig. 8 shows the results obtained with special enrichment functions incorporated into the finite element approximation. Again, only three elements across the entire bar length were used, with the same integration scheme as before. The enrichment (applied only in the middle element) was based on truncated harmonic functions

$$\hat{G}_k(\xi) = \chi(\xi;\xi_L,\xi_R)\ \sin\frac{k\pi(\xi-\xi_L)}{\xi_R-\xi_L}, \qquad k=1,2,\dots M \tag{1}$$

where ξ is the natural element coordinate varying between -1 and 1, ξ_L and ξ_R are parameters defining the position of the localization zone with respect to the element, and $\chi(\xi;\xi_L,\xi_R)$ is the characteristic function of the interval $[\xi_L,\xi_R]$, equal to 1 for ξ inside that interval and 0 for all other ξ. Functions $\hat{G}_k(\xi)$ are harmonic "pulses" added to the strain interpolation using the enhanced assumed strain technique [13]. To satisfy the compatibility condition, they are first transformed into functions

$$\begin{aligned} G_k(\xi) &= \hat{G}_k(\xi) - \frac{1}{2}\int_{-1}^{1}\hat{G}_k(\xi)\,\mathrm{d}\xi = \qquad (2)\\ &= \chi(\xi;\xi_L,\xi_R)\ \sin\frac{k\pi(\xi-\xi_L)}{\xi_R-\xi_L} - \frac{1-(-1)^k}{2k\pi}(\xi_R-\xi_L), \qquad k=1,2,\dots M \end{aligned}$$

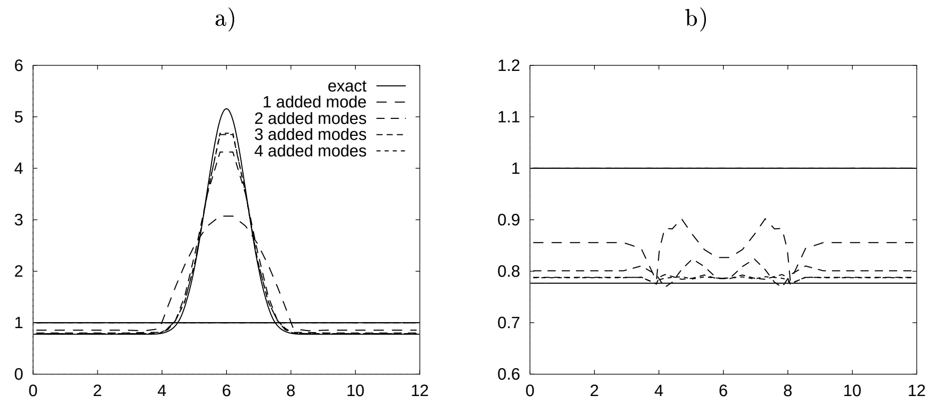

Fig. 8. Localization in a one-dimensional nonlocal continuum approximated by enriched finite elements: a) strain profiles, b) stress profiles

that have zero mean on the interval $[-1, 1]$.

The results in Fig. 8 indicate that already one added mode ($M = 1$) leads to a strain approximation comparable to cubic or quartic elements and, at the same time, the stress oscillations are substantially reduced. With three added modes ($M = 3$), the strain profile is approximated very closely and the stress oscillations become almost negligible. In fact, the agreement with the exact strain profile is better than it appears in the figure, where the values at integration points are connected by straight segments. The actual approximation has a sinusoidal shape and its value at the center of the localization zone is above the chord shown in Fig. 8a, much closer to the peak of the exact profile. Also, note that each added strain mode corresponds to one additional degree of freedom. So the complete model with three added strain modes in the middle element has as few as 5 degrees of freedom, while the finest mesh consisting of standard linear elements needed 95 degrees of freedom (or at least 33 degrees of freedom, if only the localization zone is refined).

3 Concluding Remarks

This paper has briefly sketched the idea of finite elements with embedded localization zones and illustrated it with a simple one-dimensional example. Of course, much additional work is needed to make the proposed concept useful in practical applications.

The most difficult step is no doubt the extension to multiple dimensions. In one dimension, the weak compatibility condition (vanishing integral of the enhanced strain mode over the element) is equivalent to the strong compatibility condition (continuity of displacements). In multiple dimensions, this is no longer the case, and the exact way in which weak compatibility is enforced has an important influence on the kinematic properties of the enriched model. For instance, the elements with embedded discontinuities that are derived in a variationally consistent manner from the stress continuity condition are inherently incapable of reproducing a clean separation with no stress transfer across a widely open crack; see [9] for a discussion.

Other issues to be addressed in the future include the optimal choice of enrichment functions and efficient integration schemes. Finally, it is desirable to develop the technique into a truly adaptive approach, in which the parameters of the enhanced modes (e.g., ξ_L and ξ_R in the one-dimensional example) are continuously adapted to the evolving shape of the localization profile. These topics are the subject of an ongoing research.

Acknowledgment

Financial support of the Swiss Commission for Technology and Innovation under grant CTI 4424.1 is gratefully acknowledged.

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